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Zero-Curvature FRW Models and Bianchi I Space-Time As Solutions of the Same Equation

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We show that the condition of isotropy of pressure in the case of Bianchi I space-time filled with a perfect fluid reduces via a suitable scale transformation to a linear second-order differential equation, which admits as particular solutions those of Friedmann, Robertson, and Walker. These particular solutions are then used for generating many new local rotational symmetry Bianchi I solutions. Some of their physical properties are then studied.

1. INTRODUCTION

The main difficulty in theory of general relativity is essentially due to the nonlinearity of the field equations; this is particularly obvious in the search for analytical solutions of Einstein's field equations. To remedy such a situation we have (Hajj-Boutros, 1984, 1985, 1986a–c; Hajj-Boutros and Sfeila, 1986) reduced the field equations to first-order Riccati equations.

In this work we have (via a suitable scale transformation) reduced in a straightforward manner the field equation to a linear second-order differential equation, in the case of Bianchi I space-time filled with a perfect fluid and possessing the local-rotational symmetry (LRS) (Kramer et al., 1980).

This linear differential equation is a result of the condition of isotropy of pressure and may be written as

$$A''/A = B''/B \tag{1}$$

where prime denotes $d/d\tau$, and A and B are the cosmic scale functions occurring as metric functions in the LRS Bianchi I space-time.

The main result for such a linear differential equation is the possibility of solutions like

$$A(\tau) = B(\tau) \tag{2}$$

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In this case the solutions are of Friedmann (1922; 1924)-Robertson (1935, 1936)-Walker (1936) type and constitute particular solutions (1).

More general solutions are those where $A(\tau) \neq B(\tau)$, i.e., the solutions of Bianchi-type (LRS) I in the Bianchi-Behr (Bianchi, 1897; Estabrook et al., 1968) classification. By more general solutions we mean solutions that are not isotropic, but only homogeneous (FRW solutions are isotropic and homogeneous space-time).

We show in the following section that solutions like (2) serve in fact to generate new solutions, which in general are of Bianchi type I.

3. FIELD EQUATIONS AND GENERATION TECHNIQUE

The metric for LRS Bianchi I space-time is of the form (MacCallum, 1979)

$$ds^2 = -dt^2 + A^2(t)\, dx^2 + B^2(t)(dy^2 + dz^2) \tag{3}$$

In the case of an energy-momentum tensor of a perfect fluid type, i.e.,

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab}, \qquad u_a u^a = -1 \tag{4}$$

where u^a is the 4-vector velocity, p the pressure, and μ the mass-energy density, the Einstein field equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} = \chi_0 T_{ab} \tag{5}$$

are written as

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = -\chi_0 p \tag{6}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}}{A} \cdot \frac{\dot{B}}{B} = -\chi_0 p \tag{7}$$

$$2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2} = \chi_0 p \tag{8}$$

where the dot denotes d/dt, and χ_0 is Einstein's gravitational constant.

Elimination of p from equations (6) and (7) gives the condition of isotropy of pressures,

$$\frac{\ddot{B}''}{B} - \frac{A''}{A} - \frac{A'B'}{AB} + \frac{B'^2}{B^2} = 0 \tag{9}$$

Making the scale transformation

$$d\tau = dt/B \tag{10}$$

in (9), we get

$$A''/A = B''/B \tag{11}$$

Equation (11), which is a linear second-order differential equation in A or B, admits as particular solution

$$A(\tau)=B(\tau) \tag{12}$$

In this case the metric (3) becomes

$$ds^2=-dt^2+A^2(\tau)(dx^2+dy^2+dz^2) \tag{13}$$

which is the zero-curvature FRW line element. We call these particular solutions

$$A(\tau)=B(\tau)=a(\tau) \tag{14}$$

It is to be noted that the metric (13) may be written in a conformally flat form, and thus, by taking into account the formulas (10) and (12) and we obtain

$$ds^2=A^2(\tau)(-d\tau^2+dx^2+dy^2+dz^2) \tag{15}$$

we note here that $A(\tau)=B(\tau)=c$, where c is a constant, is also a particular solution to (11) and in this case (3) is reduced to

$$ds^2=-dt^2+c^2(dx^2+dy^2+dz^2) \tag{16}$$

which is in fact a flat metric.

Going back now to the formula (11) and seeking for solutions like

$$A(\tau)=a(\tau)A_1(\tau) \tag{17}$$

or

$$B(\tau)=a(\tau)B_1(\tau) \tag{18}$$

we obtain successively

$$A(\tau)=a(\tau)\left(\int c_1\frac{d\tau}{a_1^2(\tau)}+c_2\right) \tag{19}$$

$$B(\tau)=a(\tau)\left(\int \frac{c_3\,d\tau}{a_1^2(\tau)}+c_4\right) \tag{20}$$

where c_1, c_2, c_3, and c_4 are constants of integration. It is obvious that $A(\tau)$ and $B(\tau)$ obtained from (19) and (20) are different from $a(\tau)$.

Therefore, starting from FRW or vacuum flat solutions our ansatz allows us to obtain more general ones.

3. GENERATED SOLUTIONS

3.1. Solutions Generated from Flat Metric

In the case of a flat metric we have

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{21}$$

The metric (21) is a particular case of a FRW space-time where $a(t) = 1$. Applying now the formulas (10) and (20), we get

$$B(t) = (c_3 t + c_4) \tag{22}$$

$A(t)$ stays invariable, i.e.,

$$A(t) = 1 \tag{23}$$

Without loss of generality we can set $c_3 = c_4 = 1$; thus we obtain

$$ds^2 = -dt^2 + dx^2 + t^2(dy^2 + dz^2) \tag{24}$$

This solution is that obtained by Tupper (1983).

For this solution the pressure p and the mass-density μ satisfy the relation

$$\mu = p = 1/(4t^2) \tag{25}$$

If we apply both formulas (19) and (20) we get

$$ds^2 = -dt^2 + t^2(dx^2 + dy^2 + dz^2) \tag{26}$$

For such a metric the equation of state is of the form

$$\mu + 3p = 0 \tag{27}$$

where

$$\chi_0 \mu = 3/t^2 \tag{28}$$

$$\chi_0 p = -1/t^2 \tag{29}$$

The dominant energy conditions of Hawking and Ellis (1973), i.e.,

$$\mu > 0, \qquad \mu + 3p > 0 \tag{30}$$

are then satisfied.

Finally, we note that a class of LRS Bianchi I solutions may be obtained by applying the formulas (19) and (20) as many times as we need, and the line element for the class of solutions obtained is of the form

$$ds^2 = -dt^2 + t^{\alpha_1}\, dx^2 + t^{\alpha_2}(dy^2 + dz^2) \tag{31}$$

where α_1 and α_2 are arbitrary integers.

Such a solution may be identified with that obtained by Dunn and Tupper (1980).

Note that the solution (24) represents a "barrel" singularity (MacCallum, 1971). For the metric (31) the singularity depends on the sign of α_1 and α_2.

3.2. Solutions Generated from That of Einstein and de Sitter

The Einstein-de Sitter (1932) solution reads

$$ds^2 = -dt^2 + t^{4/3}(dx^2 + dy^2 + dz^2) \tag{32}$$

which is usually interpreted as a zero-pressure, perfect fluid model.

Applying now formulas (10) and (19), we obtain

$$A(t) = t^{2/3}(-c_1/t + c_2) \tag{33}$$

where c_1 and c_2 are two constants of integration; $B(t)$ stays invariable, i.e.,

$$B(t) = t^{2/3} \tag{34}$$

and the LRS Bianchi I solution reads

$$ds^2 = -dt^2 + t^{4/3}(-c_1/t + c_2)^2\, dx^2 + t^{4/3}(dy^2 + dz^2) \tag{35}$$

Inserting now the values of $A(t)$ and $B(t)$ obtained from the formulas (33) and (34) into the formulas (6) and (8), we obtain

$$p = 0 \tag{36}$$

$$\chi_0 \mu = \frac{4}{3t}\left(\frac{2}{3t} + \frac{c_1}{-c_1 t + c_2 t^2}\right) \tag{37}$$

μ is positive for

$$c_1, c_2 > 0 \tag{38}$$

$$t \geq c_1/c_2 \tag{39}$$

Hence, we obtain a realistic physical model if the values of t are bounded by the inegalities (38) and (39). Furthermore, the shear tensor σ_{ij} has the components (Hajj-Boutros, 1086b)

$$\sigma_{11} = \frac{2}{3} A^2\left(\frac{A'}{A} - \frac{B'}{B}\right) = t^{4/3}\left(-\frac{c_1}{t} + c_2\right)^2\left(\frac{c_1}{-c_1 t + t^2}\right) \tag{40}$$

$$\sigma_{22} = \sigma_{33} = \frac{B^2}{3}\left(\frac{B'}{A} - \frac{A'}{A}\right) = t^{4/3}\left(\frac{-c_1}{-c_1 t + t^2}\right) \tag{41}$$

$$\sigma_{ij} = 0 \qquad \text{for all other } i \text{ and } j$$

and the σ_{ij} become zero when $t \to \infty$ [with $A(t) = c_2 B(t)$ when $t \to \infty$].

3.3. Solutions Generated from That of Tolman

The radiation $\mu = 3p$ solution of Tolman (1934) reads

$$ds^2 = -dt^2 + t(dx^2 + dy^2 + dz^2) \tag{42}$$

which belongs to the FRW class of solutions.

Applying again the formulas (10) and (19), we obtain

$$A(t) = t^{1/2}(-2c_1 t^{-1/2} + c_2) \tag{43}$$

$B(t)$ stays invariable, i.e.,

$$B(t) = t^{1/2} \tag{44}$$

The new LRS Bianchi I solution reads

$$ds^2 = -dt^2 + t(-2c_1 t^{-1/2} + c_2)^2\, dx^2 + t(dy^2 + dz^2) \tag{45}$$

Inserting now the new values of $A(t)$ and $B(t)$ into the formulas (6) and (8), we obtain

$$\chi_0 p = \frac{1}{4t^2} \tag{46}$$

$$\chi_0 \mu = \frac{1}{4t^2}\left(\frac{-2c_1 t^{-1/2}}{-2c_1 t^{-1/2} + c_2}\right) \tag{47}$$

For $c_2 = 0$ we obtain a stiff matter solution

$$\mu = p \tag{48}$$

For $c_1 c_2 > 0$ and $\sqrt{t} > 3c_2/2c_1$ the mass energy density μ is always bounded by the double inegality $1/(4t^2) \leq \mu \leq 3/(4t^2)$; hence, the strong energy conditions of Hawking and Ellis, i.e.,

$$\mu > 0 \tag{49}$$

$$-\mu \leq p \leq \mu \tag{50}$$

are always satisfied.

The equation of state is of the form

$$\mu = p\frac{-2c_1(1/4\chi_0 p)^{-1/4} + 3c_2}{-2c_1(1/4\chi_0 p)^{-1/4} + c_2} \tag{51}$$

Hence it is not of the form

$$p = (\gamma - 1)\mu \qquad (1 \leq \gamma \leq 2)$$

and this solution belongs to the rare solutions of this type existing in the literature.

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